Computing Asymptotic Merit Factors with Language C

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Abstract

Finding binary sequences with high merit factor has important applications in digital communications, theoretical physics, and theoretical mathematics. In this paper, we will introduce the history of Merit Factor Problem, and then present some interesting numerical results of certain families of sequences obtaining high asymptotic merit factors.

1 Introduction

The merit factor problems related to the binary sequences is an old problem proposed by Golay ([1]) in 1972. In this paper we will start with introducing definitions of binary sequences and Merit Factor Problem. Then we will use Language C to compute merit factors of certain families of binary sequences. College students who have taken College Algebra should be able to understand most content of this paper. Students having taken Calculus II, Introduction to Number Theory, and fundamental training on any computer language are supposed to read though the entire paper with no difficulties. The authors sincerely wish this paper could help more interested readers recognize such an exciting research area. Studying merit factor problems by computer searching may be a suitable research problem for college or even advanced high school students. However, the impact of the results could be very important and far- reaching.

Binary sequences are sequences with binary entries 0 and 1. For instance, $(1, 0, 1, 1, 0, \ldots)$ is a binary sequence. To avoid the occurrence of frequent and uninformative zeroes in calculations, we transfer the binary sequences to the equivalent forms via $1 = (-1)^0$ and $-1 = (-1)^1$. After this transformation, the last binary sequence becomes an equivalent one $(-1, 1, -1, -1, 1, \ldots)$. Now let $x = (x_0, x_1, \ldots, x_{N-1})$ and $y = (y_0, y_1, \ldots, y_{N-1})$ be binary sequences (with ± 1 entries) of *length* N. The *aperiodic cross-correlation* function between x and y at shift i is defined to be

$$A_{x,y}(i) = \sum_{j=0}^{N-i-1} x_j y_{j+i}\,, \quad i = 1, ..., N-1\,. \tag{1}$$

When $x = y$, denote

$$A_x(i) = A_{x,x}(i) = \sum_{j=0}^{N-i-1} x_j x_{j+i}\,, \quad i = 1, ..., N-1\,, \tag{2}$$

the *aperiodic autocorrelation* function of x at shift i.

The *periodic cross-correlation* function between x and y at shift i is defined to be

$$P_{x,y}(i) = \sum_{j=0}^{N-1} x_j\, y_{j+i}, \quad i = 0, ..., N-1\,, \tag{3}$$

Similarly, when $x = y$, the *periodic autocorrelation* function of x at shift i is

$$P_x(i) = \sum_{j=0}^{N-1} x_j\, x_{j+i}, \quad i = 0, ..., N-1\,, \tag{4}$$

Note that in all the equations above, the subscripts of the entries are taken modulo the sequence length N. For instance, if $x = (x_0, x_1, \ldots, x_{N-1}) = (-1, 1, -1, -1, 1)$, then $N = 5$, we have

$$\begin{aligned} A_x(2) &= x_0x_2 + x_1x_3 + x_2x_4 = (-1)\times(-1) + 1\times(-1) + (-1)\times 1 = -1 \\ P_x(2) &= x_0x_2 + x_1x_3 + x_2x_4 + x_3x_0 + x_4x_1 \\ &= (-1)\times(-1) + 1\times(-1) + (-1)\times 1 + (-1)\times(-1) + 1\times 1 = 1 \end{aligned}$$

If the sequence x is binary, which means that all the x_j's are $+1$ or -1, the *merit factor* of the sequence x, introduced by Golay [2], is defined as

$$F(x) = \frac{N^2}{2\sum_{i=1}^{N-1} A_x^2(i)}\,. \tag{5}$$

Moreover, let X_N be the set of all binary sequences of length N. We define F_N to be the largest merit factor of all binary sequences in X_N:

$$F_N := max_{x \in X_N} F(x)$$

Then the **Merit Factor Problem** is to determine the value of $\limsup_{N\to\infty} F_N$.

The Merit Factor Problem has wide applications in area of digital communication engineering. To see that, we look at a simplified communication system with only one signal sender and receiver. In order to communicate information from a sender to a receiver, the sender needs to transmit messages through the so called *communication channel*. It is important to distinguish here between a signal and a message. In this paper, a signal is a single digit 1 or -1, but a message means a binary sequence with entries 1 or -1.

As shown in Figure 1, in the real world, the sender and receiver have to communicate over a noisy communication channel. That is, the signals that are received may not be identical to the signals that

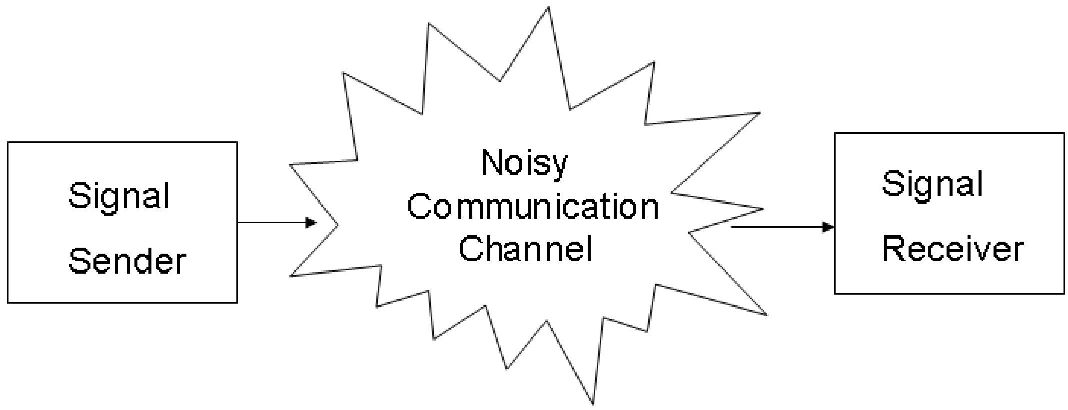

Figure 1: A Simple Model of Communication Channel

are sent. As a result, the receiver must decide which signal was actually sent, given the actual signal that was received.

If a message x is represented by a binary sequence $x = (x_0, x_1, \ldots, x_{N-1})$ of length N, we use notation $x^{[i]} = (0, \ldots, 0, x_0, x_1, \ldots, x_{N-1-i})$ to represent the sequence delayed by i time units. Then $A_x(i)$ represents the correlation between message x and the delayed message $x^{[i]}$. In communication engineering, it has been well known that using binary sequences x with a large difference between $A_x(0)$, the correlation of message x to itself, and $A_x(i)$, the correlation of message x to its delay $x^{[i]}$, can evidently reduce the detection error of messages transferred through a noisy communication channel. Note that $A_x(0) = N$ is exactly the length of the sequence x. So if we use the merit factor formula as defined in equation (5), it is equivalent to say that using binary sequences with high merit factor will significantly improve the efficiency of the communication system.

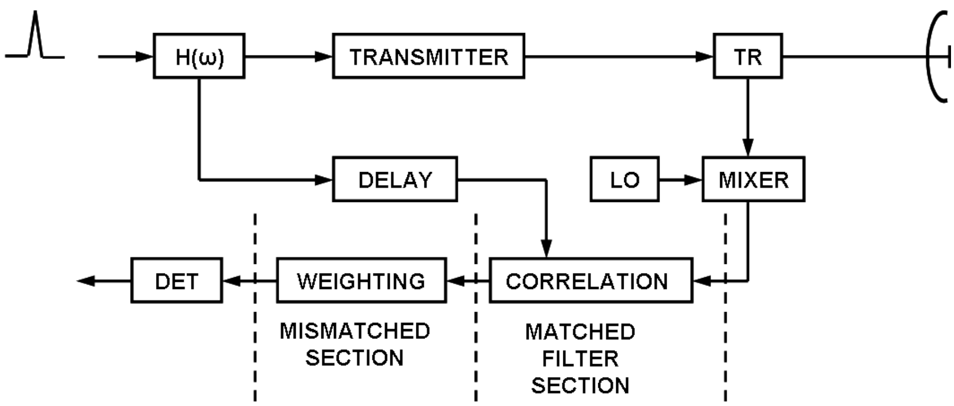

Figure 2: Model of Pulse Compression Radar System

Figure 2 is from Radar Handbook ([3], page 10.2, figure (c)). It gives a concrete example of the application of merit factor problem in a radar system. In Figure 2, when the receiver receives a signal x', it calculates all the correlations of between x' and all the delays of $x^{[i]}$ by i time units (they are exactly all aperiodic cross-correlations $A_x(i)$s). The message x having the highest correlation values with x' will be detected as the message that was actually sent. Therefore we want the correlations between x' and all the delays of $x^{[i]}$ ($A_x(i)$s) to be small, for $i > 0$, since $x^{[i]}$ is not the correct message. In this model, we can see that using binary sequences with high merit factors will obviously enhance the reliability of the radar message detector.

Merit Factor also has wide applications in theoretical physics, theoretical chemistry, several math branches like complex analysis, combinatorics and information theory. Interested readers can find more complete introduction to the history and applications of the merit factor problem from [4] and [5].

After realizing the importance of merit factor, now we need some new definitions which will be used for the rest of this paper. Given a binary sequence $X = (x_0, x_1, \ldots, x_{N-1})$ of length N, for real numbers r and t with $r, t \in [0, 1]$, we will write $X_r = (b_0, b_1, \ldots, b_{N-1})$ obtained by rotating the sequence X by a multiple r of its length:

$$b := x_{(i+\lfloor rN \rfloor) \bmod N} \ \text{ for } \ 0 \le i \le N-1$$

and will write X^t for the sequence $(c_0, c_1, \ldots, c_{\lfloor tN \rfloor - 1})$ obtained by *truncating* the sequence X to a fraction t of its length:

$$c_i := a_i \ \text{ for } \ 0 \le i \le \lfloor tN \rfloor - 1$$

Following these notations, sequence X_r^t represents the $t-$th ratio truncation of sequence X_r which is a $r-$rotation of sequence X. Moreover, for two binary sequences X and Y of lengths m and n, we use the notation $(X; Y)$ to represent the new sequence of length $m + n$ obtained by appending sequence Y at the end of sequence X.

Example 1 *Let* $N = 11$*,* $X = (+1, +1, -1, +1, +1, +1, -1, -1, -1, +1, -1)$ *is of length* $N = 11$*,* $r = 0.2$*,* $t = 0.32$*. Then* $\lfloor rN \rfloor = 2$*, so* X_r *is obtained by rotating sequence* X *by* 2 *positions:*

$$X_r = (-1, +1, +1, +1, -1, -1, -1, +1, -1, +1, +1);$$

$\lfloor tN \rfloor = 3$*, thus* X^t *is obtained by just keeping the first three digits of sequence* X*:*

$$X^t = (+1, +1, -1) \text{ and } X_r^t = (-1, +1, +1)$$

so

$$(\ X_r; X^t\ ) = (-1, +1, +1, +1, -1, -1, -1, +1, -1, +1, +1, +1, +1, -1)$$

and

$$(\ X_r; X_r^t\ ) = (-1, +1, +1, +1, -1, -1, -1, +1, -1, +1, +1, -1, +1, +1)$$

are both new sequences of length $11 + 3 = 14$*.*

Definition 2 *For* p *a given odd prime, we define* $Lp = (x_0, x_1, x_2, \ldots, x_{p-1})$ *the Legendre sequence of length* p *as*

$$x_i = \left(\frac{i}{p}\right), \quad \text{where} \left(\frac{i}{p}\right) = \begin{cases} 1 & \text{, if } i \text{ is a quadratic residue modulo } p\,; \\ -1 & \text{, otherwise.} \end{cases} \tag{6}$$

here $\left(\frac{i}{p}\right)$ *is called Legendre symbols.*

Example 3 *Let $p = 7$, we list all the position numbers i such that i is a quadratic residue modulo 7:*

$$\begin{array}{ll} \textit{if } i = 0, \ \textit{then } i \equiv 0^2 \pmod 7 & \textit{if } i = 1, \ \textit{then } i \equiv 1^2 \pmod 7 \\ \textit{if } i = 2, \ \textit{then } i \equiv 3^2 \pmod 7 & \textit{if } i = 4, \ \textit{then } i \equiv 2^2 \pmod 7 \end{array}$$

So $\left(\frac{0}{7}\right) = \left(\frac{1}{7}\right) = \left(\frac{2}{7}\right) = \left(\frac{4}{7}\right) = 1$. Therefore, $x_0 = x_1 = x_2 = x_4 = 1$, all the other x_i's are -1. And the Legendre sequence of length 7 is $L7 = (\ 1, 1, 1, -1, 1, -1, -1\ )$.

In 1988, Høholdt and Jensen calculated the asymptotic merit factors of all rotations of L_p of odd prime length p ([6]). Based on their calculations, the highest asymptotic merit factor value is 6.0 when the rotation ratio $r = 0.25$. This important discovery has inspired a large volume of results about achieving the high asymptotic merit factor 6.0 of binary sequences of length p([6]), pq([7], [8]), $2p$ ([9], [10]), $4p$([11]), $2pq$ ([12]), here p and q are distinct odd primes. However, series of discoveries ([10], [13], [5]) have revealed that certain methods can be used to construct binary sequences with asymptotic merit factor > 6.0.

In 2004, P. Borwein, Choi, and Jedwab ([13]) published the following far-reaching numerical observations:

Proposition 4 *Let L_p be a Legendre sequence of prime length p as defined in Definition 2. Then the maximum of $\lim_{p\to\infty} F((L_p)_r; (L_p)_r^t)$ is*

$$\lim_{p\to\infty} F((L_p)_r; (L_p)_r^t) \approx 6.3421$$

where the rotation ration $r \approx 0.2211$ and the appending ratio $r \approx 0.0578$.

In next section, we will construct some new families of binary sequences with high asymptotic merit factor > 6.0. We will use Language C to finish all the computations. To the authors' knowledge, most tables and figures in Section 2 have not been presented anywhere else.

2 Construction

Definition 5 *For $\delta = 0, 1$, let $\epsilon = \{\epsilon_0, \epsilon_1, \epsilon_2, \ldots, \epsilon_{N-1}\}$ be one of the four sequences $\pm\epsilon^{(\delta)}$ given by*

$$\epsilon_j^{(\delta)} = (-1)^{\binom{j+\delta}{2}} \tag{7}$$

For instance,

$$\epsilon^{(0)} = (\ +1, +1, -1, -1, \ldots, +1, +1, -1, -1, \ldots\ )$$

and

$$\epsilon^{(1)} = (\ +1, -1, -1, +1, \ldots, +1, -1, -1, +1, \ldots\ )$$

Definition 6 *Given two binary sequences $\alpha = (\ \alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{N-1}\ )$ and $\beta = (\ \beta_0, \beta_1, \ldots, \beta_{N-1}\ )$, we define sequence $b = \alpha * \beta$ by $b_i = \alpha_i \times \beta_i$, for $i = 0, 1, ..., N-1$.*

Example 7 *Using the Legendre sequence $L7$ of length $p = 7$ as in Example 3, and let the sequence $\epsilon^{(0)}$ be as defined in Definition 5, then*

$$L7 * \epsilon^{(0)} = (\ +1, +1, -1, +1, +1, -1, +1\ )$$

Given a binary sequence $\alpha = (\ \alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{N-1}\ )$, we use $-\alpha$ to represent the sequence $(\ -\alpha_0, -\alpha_1, -\alpha_2, \ldots, -\alpha_{N-1}\ )$. For instance, $-L7 = (-1, -1, -1, +1, -1, +1, +1)$.

Definition 8 *For a given odd prime p, let Lp be the Legendre sequence of length p as defined in equation (6). And $\epsilon^{(\delta)}$ is a sequence as defined in Definition 5, where $\delta = 0, 1$. Define $B2P = (Lp; Lp)*\epsilon^{(\delta)}$ as a binary sequence of length $2p$. Then using the notation defined before, sequence $B2P^{\,t}$ represents the $t-$truncation of sequence $B2P$, and $B2P_r^{\,t}$ represents the $t-$truncation of sequence $B2P_r$ which is a $r-$rotation of sequence $B2P$.*

Example 9 *For the Legendre sequence $L7 = (\ 1, 1, 1, -1, 1, -1, -1\ )$ of length $p = 7$ as in Example 3, $B2P = (L7; L7) * \epsilon^0 = (\ +1, +1, -1, +1, +1, -1, +1, -1, +1, +1, +1, -1, -1, -1\ )$. If we choose that both the rotation and truncation ratios to be $r = t = 0.3$, then*

$$B2P_{0.3} = (\ -1, -1, +1, +1, -1, +1, +1, -1, +1, -1, +1, +1, +1, -1\ )$$

and $B2P_{0.3}^{0.3} = (\ -1, -1\ )$ for $\lfloor rp \rfloor = \lfloor tp \rfloor = 2$.

Since our goal is to find binary sequences with asymptotic merit factor > 6.0, the authors have used Language C to conduct the sieve of sequences $(B2P_r;\ -B2P_r^{\,t})$ for different combinations of p, r and t values. Here please note that $-B2P_r^{\,t}$ means that if we do the rotation on sequence $B2P$ using rotation ratio r, truncate the rotated sequence $B2P_r^{\,t}$ with a truncation ratio t, and then change every entry of the new sequence $B2P_r^{\,t}$ to its negative value.

Table 1: For $p = 9973$, the Merit Factors of $(B2P_r;\ -B2P_r^{\,t})$ for Different r and t Values

		t									
		0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	...
r	0	5.9865	6.0809	6.1473	6.1793	6.1767	6.1399	6.0781	5.9858	5.8745	...
	0.01	4.8754	4.9108	4.9140	4.8999	4.8585	4.8016	4.7395	4.6545	4.5621	...
	0.02	4.1573	4.1519	4.1328	4.0947	4.0484	3.3992	3.9284	3.8521	3.7646	...
	0.03	3.6610	3.6399	3.5980	3.4489	3.5159	3.4515	3.3787	3.3089	3.2409	...
	0.04	3.3038	3.2664	3.2327	3.1964	3.1376	3.0720	3.0126	2.9441	2.8745	...
	0.05	3.0374	2.9986	2.9607	2.9013	2.8378	2.7783	2.7259	2.6701	2.5989	...
	0.06	2.8282	2.7858	2.727	2.6664	2.6110	2.5661	2.5150	2.4484	2.3903	...
	0.07	2.6556	2.5990	2.5406	2.4848	2.4463	2.3973	2.3337	2.2800	2.2231	...
	0.08	2.5310	2.4704	2.4127	2.3705	2.3256	2.2697	2.2195	2.1632	2.1101	...
	...	...	...	...	...	...	...	...	...	...	...

Based on our construction, Table 1 shows that if we choose a large $p = 9973$, then the merit factor reaches its highest value when $r = 0$ and $t \approx 0.03$. So we use a finer grid to search for t values when r is fixed to be 0. Figure 3 is the 3D-plot of the data values shown in Table 1. Figure 4 shows that when p values approach to infinity, the asymptotic merit factor of sequences $(B2P_r;\ -B2P_r^{\,t})$ obtains the

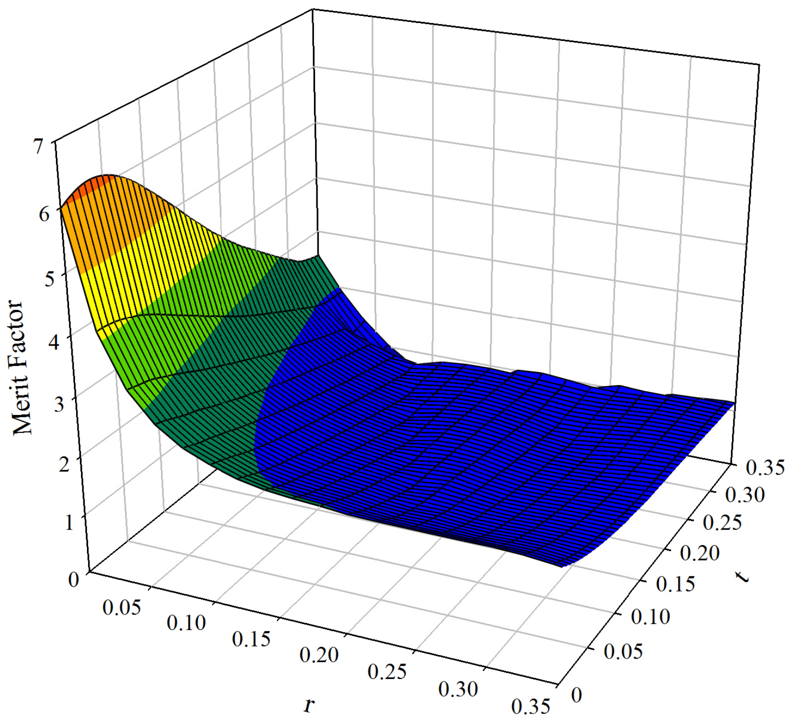

Figure 3: 3D Plot of data in Table 1

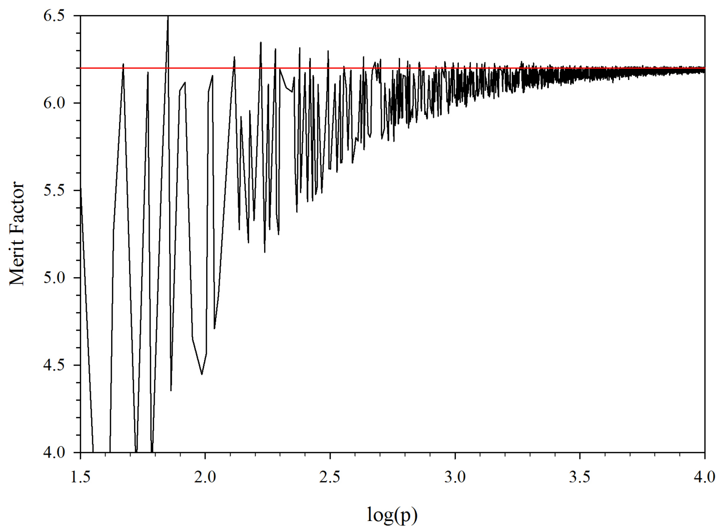

Figure 4: The Asymptotic Merit Factor of $(B2P_r;\ -B2P_r^{\ t})$ is 6.20 for $r = 0$ and $t \approx 0.026$

peak value 6.20 at $r = 0$ and $t \approx 0.026$. The C program used to obtain the results provided in Table 1, Figure 3 and Figure 4 can be found in file **sequence2p.c**.

Next interesting result of this paper will use Jacobi symbols which are generalized forms of Legendre symbols. That is, if $N = pq$, where p and q are different primes, the Jacobi symbol $\left[\frac{j}{N}\right]$ is defined by

$$\left[\frac{j}{N}\right] = \left(\frac{j}{p}\right) \cdot \left(\frac{j}{q}\right) \quad (8)$$

Where the terms on the right-hand side are the Legendre symbols as defined before. And a Jacobi sequence **z** of length $N = pq$ is defined as $z_j = \left[\frac{j}{N}\right]$.
A *modified Jacobi sequence* **m** $= \{m_0, m_1, m_{N-1}\}$ of length $N = pq$, where p and q are different primes, is defined as

$$m_j = \begin{cases} +1, & j = p, 2p, 3p, \ldots, (q-1)p; \\ -1, & j = 0, q, 2q, \ldots, (p-1)q; \\ \left[\frac{j}{N}\right], & gcd(j, N) = 1. \end{cases} \quad (9)$$

Example 10 *Suppose* $p = 3$*,* $q = 5$*, then a Legendre sequence of length* 3 *is* $L3 = (1, 1, -1)$*, and a Legendre sequence of length* 5 *is* $L5 = (1, 1, -1, -1, 1)$*, a Jacobi sequence of length* $N = pq = 15$ *is* $Z15 = (1, 1, 1, -1, 1, -1, 1, -1, 1, 1, 1, -1, -1, -1, -1)$*, and a modified Jacobi sequence of length* $N = 15$ *is* $m = (1, 1, 1, 1, 1, -1, 1, -1, 1, 1, -1, -1, 1, -1, -1)$.

In 2004, Borwein, Choi, and Jedwab have computed some asymptotic merit factor values of modified Jacobi sequences ([13]). However, they only tried fours pairs of p and q values for there are too many possible combinations of p and q to exhaust. Therefore, they did not include the asymptotic behavior of the merit factors when p and q increase. Moreover, they did not give specific values for rotation ratio r and truncation ratio t. Our computation in the following will solve these remaining problems.

In order to show the asymptotic behavior of merit factor values clearly, we have used the following method to overcome the difficulties of choosing p and q values: For each odd prime number p, we choose q to be the next prime greater than p. Then we study the merit factor of sequence $(m_r;\ m_r^t)$. It turns out that the asymptotic merit factor exceeds 6.34 quickly when p increases.

Table 2 shows that for the modified Jacobi sequences, the asymptotic merit factor reaches the peak value > 6.34 around $r \approx 0.22$ and $t \approx 0.06$. Therefore we use a finer grid in area $0.21 \leq r \leq 0.23$ and $0.05 \leq t \leq 0.07$ to search for the highest merit factor value of sequences $(m_r;\ m_r^t)$. Our computation shows that if the appending ratio t is fixed to be 0.059, the asymptotic merit factor of sequences$(m_r;\ m_r^t)$ will converge to $\approx\ 6.342$ stably before p increases to 500. 3D Figure 5 also illustrates that the highest peak of merit factors are achieved when the ration ratio $r \approx 0.221$.

To see the statement above more clearly, we fix $r = 0.221$, $t = 0.059$, and plot the merit factors of sequences $(m_r;\ m_r^t)$ as a 2D picture in Figure 6. It is interesting to see that as p increases, one branch of merit factors approaches to ≈ 6.34 much faster than another branch. Borwein, Choi, and Jedwab

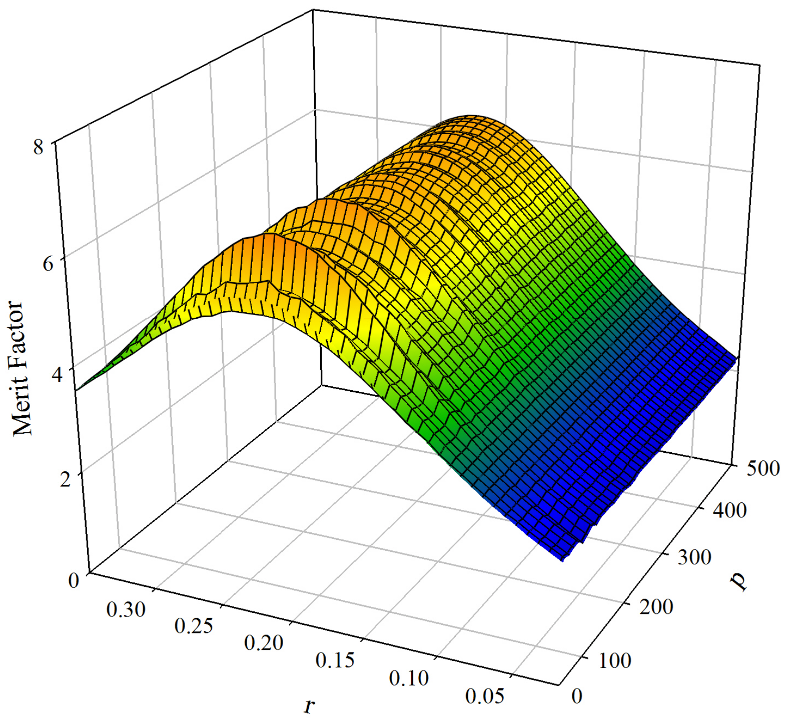

Figure 5: 3D Plot of Sequences $(m_r;\ m_r^t)$ when $t = 0.059$

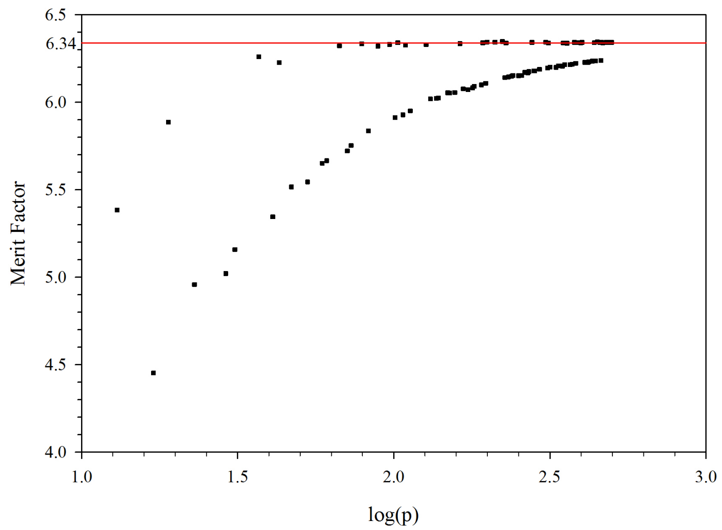

Figure 6: 2D Plot of Sequences $(m_r;\ m_r^t)$ when $r = 0.221$ and $t = 0.059$

Table 2: For $p = 491$, the Merit Factors of $(m_r;\ m_r^t)$ for Different r and t Values

		t							
		0	0.01	0.02	0.03	0.04	0.05	0.06	...
r	0	1.5000	1.5761	1.6548	1.7358	1.8189	1.9038	1.9907	...
	0.02	1.6952	1.7834	1.8743	1.9677	2.0633	2.1608	2.2596	...
	0.04	1.9250	2.0273	2.1326	2.2404	2.3504	2.4622	2.5752	...
	0.06	2.1954	2.3142	2.4361	2.5603	3.0777	3.2215	2.9415	...
	0.08	2.5132	2.610	2.7915	2.9340	2.6865	2.8137	3.3638	...
	0.10	2.8844	3.0433	3.2045	3.3666	3.5282	3.6877	3.8432	...
	0.12	3.3125	3.4937	3.6753	3.8557	4.0326	4.2049	4.3699	...
	0.14	3.7953	3.9974	4.1968	4.3915	4.5796	4.7558	4.9190	...
	0.16	4.3200	4.5381	4.7491	4.9490	5.1348	5.3031	5.4536	...
	0.18	4.8570	5.0802	5.2894	5.4806	5.6504	5.7973	5.9145	...
	0.20	5.3566	5.5690	5.7590	5.9225	6.0566	6.1600	6.2286	...
	0.21	5.5716	5.7705	5.9426	6.0847	6.1949	6.2708	6.3121	...
	0.22	5.7511	5.9316	6.0820	6.1998	6.2831	6.3297	6.3415	...
	0.23	5.8866	6.0440	6.1687	6.2584	6.3108	6.3288	6.3140	...
	0.24	5.9709	6.1020	6.1980	6.2572	6.2808	6.2719	6.2290	...
	...	...	...	...	...	...	...	...	...

claimed in their paper ([13], page 18) that the merit factor of $(m_r;\ m_r^t)$ converges to ≈ 6.34 faster when $p \equiv q \pmod 4$ without providing sufficient numerical results to support their claim. While Figure 6 gives a straightforward visual witness to their claim since the top branch represents exactly the cases that $p \equiv q \pmod 4$. The C program used to obtain the results provided in Table 2, Figure 5 and Figure 6 can be found in file **sequencepq.c**

Remark As we have discussed at the beginning of this paper, looking for binary sequences with merit factor has important applications in real life and scientific research areas. However, randomly searching binary sequences with ideal merit factors is undoubtedly impractical. Figures 3, 4, 5 and 6 show that there exist reliable algorithms that can generate binary sequences with high merit factors. And the merit factors of the generated sequences increase robustly quickly when the length of sequences increase.

3 Conclusion

In this paper we have shown some "large " asymptotic merit factors (> 6.0) by exploring certain binary sequences using Language C. In a very recent paper, Jedwab, Katza, and Schmidta have theoretically proven that ≈ 6.342 is the largest asymptotic merit factor of known binary sequences [5]. Therefore, any discovery of binary sequences with asymptotic merit factor exceeding 6.342 will be an important break through on Merit Factor problem. And it is evident that approaching this problem by computer searching can be conducted by college or even some advanced high school students.

4 Acknowledgement

The authors want to express their deepest gratitude to Dr. Xuejian Liu for his extremely professional contribution on the C programming used in this paper!

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5 Electronic Supplemental Materials

1. **sequence2p.c**
2. **sequencepq.c**